Abstract

I start from the Bargmann-Wigner equations and introduce an interaction in the form which is similar to a $j = 1/2$ case [M. Moshinsky & A. Szczepaniak, *J. Phys.* A**22** (1989) L817]. By means of the expansion of the wave function in the complete set of γ-matrices one can obtain the equations for a system which could be named as the $j = 0$ Kemmer-Dirac oscillator.

The equations for the components ϕ_1 and ϕ_2 are different from the ones obtained by Y. Nedjadi & R. Barrett for the $j = 0$ Duffin-Kemmer-Petiau (DKP) oscillator [*J. Phys.* A**27** (1994) 4301]. This fact leads to the dissimilar energy spectrum of the $j = 0$ relativistic oscillator.

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More about the $j = 0$ relativistic oscillator* †

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Meantime the problem of interaction of a spinor particle with external fields is understood well, one can not say that on interactions of bosons and higher spin fermions [1]. In the present article I consider the oscillatorlike interaction of a $j = 0$ relativistic particle in the formalism firstly introduced by Kemmer [2]-[5]. The problem is shown to be exactly solvable.

A general system of relativistic wave equations for an arbitrary spin was firstly written by Dirac [3] and Fierz [6]. In my presentation I use a reformulation of their formalism by Bargmann and Wigner [7]. For the cases of spin-0 and spin-1 the Bargmann-Wigner set[1] reduces to the two equations which can be written in the form (e. g., ref. [8])

$$\begin{cases} \left[i\gamma^\mu \partial_\mu - m\right] \Psi(x) = 0 \\ \\ \Psi(x) \left[i(\gamma^\mu)^T \partial_\mu - m\right] = 0, \end{cases} \tag{1}$$

where the wave function presents oneself a 4×4 matrix (symmetric in a $j = 1$ case and antisymmetric in a $j = 0$ case) and the derivative acts to the left hand in the second equation.

Let introduce the interaction like a $j = 1/2$ case, ref. [9],

$$\left[i\gamma^\mu \partial_\mu - k\gamma^i\gamma^0 r^i - m\right] \Psi(x) = 0, \tag{2}$$

$$\Psi(x) \left[i(\gamma^\mu)^T \partial_\mu - k(\gamma^i\gamma^0)^T r^i - m\right] = 0. \tag{3}$$

Then, let expand Ψ in terms of a set of 16 γ- matrices. It is possible to assure ourselves that the symmetry of the wave function is preserved in the expansion if the set is chosen as C, $\gamma^5 C$, $\gamma^5\gamma^\mu C$, providing the antisymmetric part, and $\gamma^\mu C$, $\sigma^{\mu\nu} C$, for the symmetric part (this form of the interaction does not mix the $j = 0$ and $j = 1$ states). C is the matrix of a charge conjugation.

By using the properties: $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$ and $C(\sigma^{\mu\nu})^T C^{-1} = -\sigma^{\mu\nu}$ in the case of the spin-0 wave function[2]

$$\Psi_{[\alpha\beta]} = C_{\alpha\beta}\varphi + \gamma^5_{\alpha\tau} C_{\tau\beta}\tilde\varphi + \gamma^5_{\alpha\delta}\gamma^\mu_{\delta\tau} C_{\tau\beta}\tilde A_\mu, \tag{4}$$

one can come to

$$\begin{cases} m\varphi = 0 \\ m\tilde\varphi = -i(\partial_\nu \tilde A^\nu) \\ m\tilde A_\mu = -i\partial_\mu\tilde\varphi + k\left[g^{0\nu} g_{\mu i} + g^{i\nu} g_{\mu 0}\right] r^i \tilde A_\nu. \end{cases} \tag{5}$$

[1]This name is usually referred to the case of a symmetric wave function, $j = 1$ and higher. However, it is easy to show that these equations (1) describe a $j = 0$ particle in the case of an antisymmetric wave function, see below.

[2]The case of the spin $j = 1$ will be reported elsewhere.

Thus, the initial reducible representation is decomposed into the $(1/2, 1/2)$ vector representation, the $(0, 0)$ scalar representation and the trivial (pseudo)scalar representation, similar to the Duffin-Kemmer-Petiau algebra.

Without interaction ($k = 0$) the above equations coincide (within the definition of κ, the constant which is proportional to mass) with Eqs. (26.12) in [4] and Eqs. $(247, 247')$ in ref. [5a], what characterizes the formalism of Kemmer[3]:

$$\begin{cases} m\tilde{\varphi} = -i\partial_\mu \tilde{A}^\mu \\ m\tilde{A}_\mu = -i\partial_\mu \tilde{\varphi}. \end{cases} \tag{6}$$

After a substitution of the second equation to the first one they yield the Klein-Gordon equation for a spinless particle.

For the stationary states $\tilde{\varphi}(x,t) = \tilde{\varphi}(x)\exp(-iEt)$, $\tilde{A}_\mu(x,t) = \tilde{A}_\mu(x)\exp(-iEt)$ the above set (5) is rewritten to[4]

$$\begin{cases} m\tilde{\varphi} = -E\tilde{A}_0 - i\nabla\vec{\tilde{A}} \\ m\tilde{A}_0 = -E\tilde{\varphi} + k(\vec{r}\vec{\tilde{A}}) \\ m\vec{\tilde{A}} = i\nabla\tilde{\varphi} + k\vec{r}\tilde{A}_0 \end{cases} \tag{7}$$

(cf. with Eqs. (9) in ref. [10]). I shall demonstrate that the above equations describe oscillatorlike system. The $j = 0$ relativistic oscillators are also considered in [11]-[13].

After simple algebraic transformations one can come to the following set of equations:

$$\begin{cases} (E - m)\phi_1 = \vec{p}^{\,-}\vec{\tilde{A}} \\ (E + m)\phi_2 = \vec{p}^{\,+}\vec{\tilde{A}} \\ \vec{p}^{\,+}\phi_1 - \vec{p}^{\,-}\phi_2 = m\vec{\tilde{A}}, \end{cases} \tag{8}$$

where $\vec{p} = \frac{1}{i}\vec{\nabla}$, $\vec{p}^{\,\pm} = \frac{1}{\sqrt{2}}(\vec{p} \pm k\vec{r})$ and

$$\phi_1 = \frac{\tilde{A}_0 - \tilde{\varphi}}{\sqrt{2}}, \qquad \phi_2 = \frac{\tilde{A}_0 + \tilde{\varphi}}{\sqrt{2}}. \tag{9}$$

Multiplying the first and the second equations by m one finds

$$\begin{cases} m(E - m)\phi_1 = \vec{p}^{\,-}\vec{p}^{\,+}\phi_1 - \vec{p}^{\,-}\vec{p}^{\,-}\phi_2 \\ \\ m(E + m)\phi_2 = \vec{p}^{\,+}\vec{p}^{\,+}\phi_1 - \vec{p}^{\,+}\vec{p}^{\,-}\phi_2, \end{cases} \tag{10}$$

[3] This formulation is also contained in the more general formulation of Dirac [3] as mentioned in ref. [4]. Therefore, I take a liberty to name the equations (15,16) as the Kemmer-Dirac oscillator.

[4] I chose a dependence of the wave function on time similar to refs. [9, 10]. If use $\tilde{\varphi}(x,t) = \tilde{\varphi}(x)\exp(iEt)$, $\tilde{A}_\mu(x,t) = \tilde{A}_\mu(x)\exp(iEt)$ the components ϕ_1 and ϕ_2 are only interchanged each other and $\omega \to -\omega$ in Eqs. (15,16), what, surprisingly, does not lead to any change of the energy spectrum.

and acting $m(E+m)$ at the first equation and $m(E-m)$ at the second one yields

$$\begin{cases} m^2(E^2-m^2)\phi_1 = m(E+m)\vec{p}^-\vec{p}^+\phi_1 - (\vec{p}^-\vec{p}^-)(\vec{p}^+\vec{p}^+)\phi_1 + (\vec{p}^-\vec{p}^-)(\vec{p}^+\vec{p}^-)\phi_2 \\ m^2(E^2-m^2)\phi_2 = -m(E-m)\vec{p}^+\vec{p}^-\phi_2 - (\vec{p}^+\vec{p}^+)(\vec{p}^-\vec{p}^-)\phi_2 + (\vec{p}^+\vec{p}^+)(\vec{p}^-\vec{p}^+)\phi_1. \end{cases} \tag{11}$$

Finally, by means of the use of the following commutation relations:

$$\left[p_i^+\, p_j^-\right]_- = ik\delta_{ij}, \qquad \left[p_i^\pm\, p_j^\pm\right]_- = 0, \tag{12}$$

$$\left\{p_i^- p_j^+ - p_i^+ p_j^-\right\} f(\vec{r}) = \left[-ik\delta_{ij} + k\epsilon_{jik}\hat{L}_k\right] f(\vec{r}), \tag{13}$$

$$\left\{\vec{p}^-\vec{p}^+ + \vec{p}^+\vec{p}^-\right\} f(\vec{r}) = \left[\vec{p}^2 - k^2\vec{r}^2\right] f(\vec{r}), \tag{14}$$

(with L_k being the operator of angular momentum and $k = im\omega$) for the $j = 0$ case we obtain

$$(E^2-m^2)\phi_1 = \left[\vec{p}^2 + m^2\omega^2\vec{r}^2 + (E+2m)\omega + \omega^2\hat{L}^2\right]\phi_1, \tag{15}$$

$$(E^2-m^2)\phi_2 = \left[\vec{p}^2 + m^2\omega^2\vec{r}^2 + (E-2m)\omega + \omega^2\hat{L}^2\right]\phi_2. \tag{16}$$

In fact, one has the oscillator-behaved term $(m^2\omega^2\vec{r}^2)$; however, there are additional terms comparing with Eq. (10) of the paper [10], the Duffin-Kemmer-Petiau oscillator. The operator of the angular momentum $\hat{L}^2$ does not present in the equations of ref. [10] and there is no a dependence of the "constant" term on the energy there. The presence of this term could lead to some speculations since one can show that a consequence of this fact is the "splitting" of energy levels in the both of equations. Namely, one has two roots in each of equations. Moreover, if pass to the nonrelativistic limit ($E = \epsilon + mc^2$, $\epsilon \ll mc^2$) in Eq. (15) one has the quantity $(2mc^2 - \hbar\omega)\epsilon$, which could be equal to zero or even negative. Meantime, the sum of the remained terms on the *rhs* in the first equation (15) is positive. Does this fact signify that the oscillator system surveys not for all frequency values? More detailed analysis presented below permits us to answer these questions.

Now let seek to solve Eq. (15). For identification purposes, in what follows it is $(E^2_{N,\ell} - m^2)/2m$ rather than $E_{N,\ell}$ which I seek since the first form reduces to the usual Schrödinger eigenvalue in the non-relativistic limit.

If use the basis functions similar to ref. [9], then $\hat{L}^2\phi_{1,2} = \ell_{1,2}(\ell_{1,2}+1)\phi_{1,2}$ and energy eigenvalues of the equation associated with Eq. (15) could be found from the algebraic equation

$$\frac{1}{2m}(E^2-m^2) - (E+2m)\frac{\omega}{2m} - \ell_1(\ell_1+1)\frac{\omega^2}{2m} = (N_1 + \frac{3}{2})\omega, \tag{17}$$

where the principal quantum number is a non-negative integer. This equation is quadratic in E and has therefore 2 roots. The solutions of equation (17) are:

$$\frac{1}{2m}(E^2_\pm - m^2) = (N_1 + \frac{5}{2})\omega + \left(\ell_1(\ell_1+1) + \frac{1}{2}\right)\frac{\omega^2}{2m} \pm \Delta_1, \tag{18}$$

where

$$\Delta_1 = \frac{\omega}{2}\left(1 + (2N_1 + 5)\left(\frac{\omega}{m}\right) + \left(\ell_1 + \frac{1}{2}\right)^2 \left(\frac{\omega}{m}\right)^2\right)^{\frac{1}{2}}. \tag{19}$$

This formula has structural similarities with the eigenvalues found for the DKP oscillator, ref. [10], *i. e.*, it involves the usual 3-dimensional harmonic oscillator energy, a term proportional to $\ell(\ell+1)$ which appears as some kind of rotational energy and a third energy contribution Δ which is a complicated function of the oscillator frequency, ℓ_1 and N_1.

In the limit where the oscillator frequencies are such that $\hbar\omega \ll mc^2$, keeping only the first-order term in ω in the equations leads to

$$\frac{1}{2m}(E_+^2 - m^2) \simeq \epsilon^+ = (N_1 + 3)\omega, \tag{20}$$

$$\frac{1}{2m}(E_-^2 - m^2) \simeq \epsilon^- = (N_1 + 2)\omega. \tag{21}$$

I now seek to solve the second equation (16). Using the same procedure as above the two eigenvalues of the energies are:

$$\frac{1}{2m}(E_\pm^2 - m^2) = (N_2 + \frac{1}{2})\omega + \left(\ell_2(\ell_2 + 1) + \frac{1}{2}\right)\frac{\omega^2}{2m} \pm \Delta_2, \tag{22}$$

where

$$\Delta_2 = \frac{\omega}{2}\left(1 + (2N_2 + 1)\left(\frac{\omega}{m}\right) + \left(\ell_2 + \frac{1}{2}\right)^2 \left(\frac{\omega}{m}\right)^2\right)^{\frac{1}{2}}. \tag{23}$$

In the limit of low frequencies

$$\frac{1}{2m}(E_+^2 - m^2) \simeq \epsilon^+ = (N_2 + 1)\omega, \tag{24}$$

$$\frac{1}{2m}(E_-^2 - m^2) \simeq \epsilon^- = N_2\,\omega. \tag{25}$$

The condition of a compatibility of the set of equations (15,16) ensures us that $N_1 = N_2 + 2$ and $\ell_2 = \ell_1$. Therefore, in the relativistic region we have two physical (positive and negative) values of the energy like to the other formulations of a interacting $j = 0$ relativistic particle. However, a remarkable feature of the presented formulation is the double degeneracy (in N) of the levels in the limit $\hbar\omega \ll mc^2$ except for the ground level. Let us note that such a phenomenon has been discovered in ref. [12] (*cf.* $\epsilon^\pm$ with Eqs. (11a,11b) of the cited work). However, the reasons of a introduction of the matrix structure in the Klein-Gordon equation have not been explained there. Next, I would like to note that the quantity $(E_\pm^2 - m^2)/2m$ is seen from Eqs. (18,19) or (22,23) to be non-negative even in the high-frequency limit.

In conclusion, let me mention that a behavior of a scalar particle in external fields has been considered in many publications, see, e. g., for the references [14, 15]. The recent publications, ref. [16], concern with a solution of the problem of finding the energy spectra of a scalar particle with a polarizability in the constant magnetic, electric fields and in the field of the plane electromagnetic wave. However, as we learnt, the model of the $j = 0$ oscillator with the intrinsic spin structure has very specific peculiarities what differs it from the other model used, e. g., in descriptions of π- and K- mesons.

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